

# Inversion of the Radon Transform on the Laguerre Hypergroup by using Generalized Wavelets

M. M. Nessibi and K. Trimèche

*Faculty of Sciences of Tunis. Department of Mathematics. University of Tunis II. 1060.*

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We consider the Radon transform  $R_\alpha$ ,  $\alpha \geq 0$ , on the Laguerre hypergroup  $K = [0, +\infty[ \times \mathbb{R}$ . We characterize a space of infinitely differentiable and rapidly decreasing functions together with their derivatives such that  $R_\alpha$  is a bijection from this space onto itself. We establish an inversion formula and a Plancherel theorem for the operator  $R_\alpha$ . Finally, by using the continuous wavelet transform on the Laguerre hypergroup  $K$ , we deduce another expression for the inverse  $R_\alpha^{-1}$  of the operator  $R_\alpha$ . © 1997 Academic Press

## INTRODUCTION

This paper deals with the Radon transform on the Laguerre hypergroup, which generalizes the Radon transform (see [17, p. 384]) of radial functions on the  $(2n + 1)$ -dimensional Heisenberg group  $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$  with the multiplication law

$$(z, t) \cdot (z', t') = (z + z', t + t' - \operatorname{Im}(z_1 \overline{z'_1} + \cdots + z_n \overline{z'_n})).$$

The unitary group  $U(\mathbb{C}^n)$  acts naturally on  $\mathbb{H}^n$  via  $u \cdot (z, t) = (u \cdot z, t)$ , for  $u \in U(\mathbb{C}^n)$ , hence  $U(\mathbb{C}^n)$  acts on functions  $F$  on  $\mathbb{H}^n$  via  $F^u(z, t) = F(u \cdot (z, t))$ .

Let  $L^1(\mathbb{H}^n)$  be the space of integrable functions on  $\mathbb{H}^n$ . It is well known that the set  $L_{rad}^1(\mathbb{H}^n) = \{F \in L^1(\mathbb{H}^n) / F^u = F \text{ for all } u \in U(\mathbb{C}^n)\}$  is a commutative subalgebra of  $L^1(\mathbb{H}^n)$  and a radial function on  $\mathbb{H}^n$  depends only on  $(\|z\|, t)$ , where  $\|z\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$  is the usual norm on  $\mathbb{C}^n$  (see [3, pp. 76–78]).

The Radon transform  $R$  on the Heisenberg group  $\mathbb{H}^n$  is defined in [17, p. 384] by

$$R(F)(z, t) = \int_{\mathbb{C}^n} F((z, t) \cdot (\xi, 0)) d\xi, \quad (1)$$

where  $d\xi$  is the Lebesgue measure on  $\mathbb{C}^n \cong \mathbb{R}^{2n}$ .

This is a particular case of the Radon transform on a general nilpotent Lie group as defined in [4].

When  $F$  is radial, i.e.,  $F(z, t) = f(\|z\|, t)$ , where  $f$  is a function defined on  $K = [0, +\infty[ \times \mathbb{R}$  the relation (1) becomes

$$R(F)(z, t) = \frac{2\pi^n}{(n-1)!} \int_0^{+\infty} T_{(\|z\|, t)}^{(n-1)} f(y, 0) y^{2n-1} dy, \quad (2)$$

where  $T_{(x, t)}^{(n-1)}$ ,  $(x, t) \in K$ , are the generalized translation operators defined for  $\alpha \geq 0$  by

$$T_{(x, t)}^{(\alpha)} f(y, s) = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} f(\sqrt{x^2 + y^2 + 2xy \cos \theta}, s + t + xy \sin \theta) d\theta, \\ \quad \text{if } \alpha = 0 \\ \frac{\alpha}{\pi} \int_0^{2\pi} \int_0^1 f(\sqrt{x^2 + y^2 + 2xyr \cos \theta}, s + t + xyr \sin \theta) \\ \quad r(1-r^2)^{\alpha-1} dr d\theta, \quad \text{if } \alpha > 0. \end{cases}$$

For  $\alpha = n-1$ ,  $n$  being a positive integer, these operators can be derived from the ordinary convolution of radial functions on  $\mathbb{H}^n$  (see [16]). More precisely, if  $F$  and  $G$  belong to  $L_{rad}^1(\mathbb{H}^n)$  such that  $F(z, t) = f(\|z\|, t)$  and  $G(z, t) = g(\|z\|, t)$  we have

$$(F * G)(z, t) = 2\pi^{n-1} \int_K T_{(\|z\|, t)}^{(n-1)} f(y, s) g(y, -s) dm_{n-1}(y, s),$$

$$(z, t) \in \mathbb{H}^n,$$

where  $*$  is the ordinary convolution product on  $\mathbb{H}^n$  and  $dm_{n-1}$  denotes the positive measure defined on  $K$ , for  $\alpha \geq 0$  by

$$dm_\alpha(x, t) = \frac{1}{\pi \Gamma(\alpha + 1)} x^{2\alpha+1} dx dt.$$

Hence  $L_{rad}^1(\mathbb{H}^n)$  can be identified with the commutative algebra  $L_{n-1}^1(K)$  of integrable functions on  $K$  with respect to the measure  $dm_{n-1}$  and with

convolution product  $*$  defined by

$$(f * g)(x, t) = \int_K T_{(x, t)}^{(n-1)} f(y, s) g(y, -s) dm_{n-1}(y, s), \quad (x, t) \in K.$$

For general real  $\alpha \geq 0$ , we provide  $K$  with a structure of the hypergroup as

$$(\mu * \nu)(f) = \int_{K \times K} T_{(x, t)}^{(\alpha)} f(y, s) d\mu(x, t) d\nu(y, s),$$

$\mu$  and  $\nu$  being bounded Radon measures on  $K$ .

For  $\alpha \geq 0$ , the Radon transform  $R_\alpha$  is defined on the hypergroup  $K$  by

$$R_\alpha f(x, t) = \frac{2\pi^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^{+\infty} T_{(x, t)}^{(\alpha)} f(y, 0) y^{2\alpha+1} dy.$$

For  $\alpha = n - 1$ ,  $n$  being a positive integer, we find the transform (2).

Some properties of the Radon transform on the Heisenberg group  $\mathbb{H}^n$  were established by D. Geller and E. M. Stein [5, 6]. They essentially proved that the operator  $(\partial/\partial t)^n R$  extends to the space  $L^2(\mathbb{H}^n)$  of square integrable functions on  $\mathbb{H}^n$  and that

$$\left\| \left( \frac{\partial}{\partial t} \right)^n R(F) \right\|_{L^2} = (4\pi)^{2n} \|F\|_{L^2}^2.$$

R. S. Strichartz [17, p. 385] observed that this formula yields the inversion formula

$$R^{-1} = \frac{1}{(4\pi)^{2n}} \left( \frac{\partial}{\partial t} \right)^n R \left( \frac{\partial}{\partial t} \right)^n$$

but he didn't specify the spaces on which this formula holds.

The purpose of this work is to establish for the Radon transform  $R_\alpha$ ,  $\alpha$  being a nonnegative number, the following results

(i) We define and characterize a subspace, denoted  $\mathcal{S}_{*,2}(K)$ , of the space of infinitely differentiable and rapidly decreasing functions together with all their derivatives on  $K$ , such that the operator  $R_\alpha$  is a bijection from  $\mathcal{S}_{*,2}(K)$  onto itself.

(ii) We give the following inversion formula for  $R_\alpha$ , when applied to the space  $\mathcal{S}_{*,2}(K)$

$$(2\pi)^{2\alpha+2} f = (L_\alpha R_\alpha)(L_\alpha R_\alpha)(f), \quad (3)$$

where  $L_\alpha$  is an integro-differential operator which coincides with  $(\partial/\partial t)^n$  for  $\alpha = n - 1$ ,  $n$  being a positive integer.

(iii) We establish a Plancherel formula for the operator  $R_\alpha$ . More precisely, we show that the operator  $(2\pi)^{-(\alpha+1)}L_\alpha R_\alpha$  extends to an isometric isomorphism from the space  $L_\alpha^2(K)$  of square integrable functions on  $K$  with respect to the positive measure  $dm_\alpha$  onto itself, and that the formula (3) holds on  $L_\alpha^2(K)$ .

Analogous results with (i), (ii), and (iii) for the classical Radon transform on  $\mathbb{R}^n$  were proved by D. Ludwig [11], S. Helgason [8], and D. C. Solmon [13].

We establish also in this paper a formula which gives the inverse of the operator  $R_\alpha$  when applied to the space  $\mathcal{S}_{*,2}(K)$ , by using generalized wavelets on  $K$  which are defined and studied in [12]. This formula is better than (3) because we have a large choice of generalized wavelets that we can use in this formula.

## I. LAGUERRE HYPERGROUP

*Notations.* We denote by

- $K = [0, +\infty[ \times \mathbb{R}$ .
- $\mathcal{C}_{*,c}(K)$  the space of continuous functions on  $\mathbb{R}^2$ , with compact support and even with respect to the first variable.
- $M_b(K)$  the space of bounded Radon measures on  $K$ .
- $L_\alpha^p(K)$ ,  $p \in [1, +\infty]$ , the space of measurable functions on  $K$  such that  $\|f\|_{\alpha,p} < +\infty$ ,

where

$$\|f\|_{\alpha,p} = \left[ \int_K |f(x,t)|^p dm_\alpha(x,t) \right]^{1/p}, \quad \text{if } p \in [1, +\infty[,$$

$$\|f\|_{\alpha,\infty} = \operatorname{ess\,sup}_{(x,t) \in K} |f(x,t)|.$$

**DEFINITION I.1.** Let  $f \in \mathcal{E}_{*,c}(K)$ . For all  $(x, t)$  and  $(y, s)$  in  $K$ , we put

$$T_{(x,t)}^{(\alpha)} f(y, s) = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} f\left(\sqrt{x^2 + y^2 + 2xy \cos \theta}, s + t + xy \sin \theta\right) d\theta, & \text{if } \alpha = 0 \\ \frac{\alpha}{\pi} \int_0^{2\pi} \int_0^1 f\left(\sqrt{x^2 + y^2 + 2xyr \cos \theta}, s + t + xy r \sin \theta\right) r(1 - r^2)^{\alpha-1} dr d\theta, & \text{if } \alpha > 0. \end{cases}$$

The operators  $T_{(x,t)}^{(\alpha)}$ ,  $(x, t) \in K$ , are called generalized translation operators on  $K$ .

**PROPOSITION I.1.** (i) Let  $f$  be in  $\mathcal{E}_{*,c}(K)$ , supported in  $[-\rho, \rho] \times [-\rho, \rho]$ . Then for all  $(x, t) \in K$ , the function  $T_{(x,t)}^{(\alpha)} f$  belongs to  $\mathcal{E}_{*,c}(K)$  and is supported in  $[x - \rho, x + \rho] \times [-t - x - x(x + \rho), -t + x + x(x + \rho)]$ .

(ii) Let  $f$  be in  $L_\alpha^p(K)$ ,  $p \in [1, +\infty]$ . Then for all  $(x, t) \in K$ , the function  $T_{(x,t)}^{(\alpha)} f$  belongs to  $L_\alpha^p(K)$  and satisfies

$$\|T_{(x,t)}^{(\alpha)} f\|_{\alpha, p} \leq \|f\|_{\alpha, p}.$$

**DEFINITION I.2.** The convolution product  $*$  on  $M_b(K)$  is defined by

$$(\mu * \nu)(f) = \int_{K \times K} T_{(x,t)}^{(\alpha)} f(y, s) d\mu(x, t) d\nu(y, s).$$

Moreover  $\mu * \nu = \nu * \mu$ , i.e., the convolution product is commutative.

**PROPOSITION I.2.** (i) Let  $\nu_1, \nu_2 \in M_b(K)$  such that  $\nu_1 = f_1 m_\alpha$  and  $\nu_2 = f_2 m_\alpha$ ,  $f_1$  and  $f_2$  being in  $L_\alpha^1(K)$ . Then we have  $(\nu_1 * \nu_2) = (f_1 * f_2) m_\alpha$ , where  $(f_1 * f_2)$  is the convolution product of  $f_1$  and  $f_2$  defined by

$$(f_1 * f_2)(x, t) = \int_K T_{(x,t)}^{(\alpha)} f_1(y, s) f_2(y, -s) dm_\alpha(y, s), \quad (x, t) \in K \quad (\text{I.1})$$

(ii) Let  $f_1 \in L_\alpha^1(K)$  and  $f_2 \in L_\alpha^p(K)$ ,  $p \in [1, +\infty]$ . Then the function  $(f_1 * f_2)$  is defined almost everywhere on  $K$  and satisfies

$$\|f_1 * f_2\|_{\alpha, p} \leq \|f_1\|_{\alpha, 1} \|f_2\|_{\alpha, p}. \quad (\text{I.2})$$

**THEOREM I.1.**  $(K, *, i)$  is a hypergroup in the sense of Jewett [1, 9], where  $i$  denotes the involution defined by  $i(x, t) = (x, -t)$ , for  $(x, t) \in K$ .

## II. GENERALIZED FOURIER TRANSFORM ON THE LAGUERRE HYPERGROUP

### 1. Laguerre Functions

We consider the following system of partial differential operators

$$\begin{cases} D_1 = \frac{\partial}{\partial t} \\ D_2 = \frac{\partial^2}{\partial x^2} + \frac{2\alpha + 1}{x} \frac{\partial}{\partial x} + x^2 \frac{\partial^2}{\partial t^2}, \end{cases} \quad (x, t) \in ]0, +\infty[ \times \mathbb{R}, \quad (\text{II.1})$$

where  $\alpha$  is a nonnegative number.

For  $\alpha = n - 1$ ,  $n$  being a positive integer, the operator  $D_2$  is the radial part of the subLaplacian on the Heisenberg group  $\mathbb{H}^n$  (see [14]).

For all  $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$  and  $(x, t) \in K$ , we put  $\varphi_{\lambda, m}(x, t) = e^{i\lambda t} \mathcal{L}_m^{(\alpha)}(|\lambda|x^2)$ , where  $\mathcal{L}_m^{(\alpha)}$  is the Laguerre function defined on  $[0, +\infty[$  by  $\mathcal{L}_m^{(\alpha)}(x) = e^{-x/2} L_m^{(\alpha)}(x) / L_m^{(\alpha)}(0)$ ,  $L_m^{(\alpha)}$  being the Laguerre polynomial of degree  $m$  and order  $\alpha$ .

**PROPOSITION II.1.** *For all  $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$ , the function  $\varphi_{\lambda, m}$  is the unique solution of the following system of partial differential operators*

$$\begin{cases} D_1 u = i\lambda u \\ D_2 u = -4|\lambda| \left( m + \frac{\alpha + 1}{2} \right) u; \\ u(0, 0) = 1, \quad \frac{\partial u}{\partial x}(0, t) = 0 \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

(See [3, p. 81], when  $\alpha = n - 1$ ,  $n$  is a positive integer.)

**Remark II.1.** (i) For  $\alpha = n - 1$ ,  $n$  being a positive integer, the functions

$$(z, t) \rightarrow \varphi_{\lambda, m}(\|z\|, t)$$

are spherical functions of the Gelfand pair  $(G, U(\mathbb{C}^n))$ , where  $G = U(\mathbb{C}^n) \ltimes \mathbb{H}^n$  is the semi-direct product of  $U(\mathbb{C}^n)$  by  $\mathbb{H}^n$  (see [3, p. 78]).

(ii) For general real  $\alpha \geq 0$ , the functions  $\varphi_{\lambda, m}$ ,  $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$ , are characters of the Laguerre hypergroup  $K$ .

**PROPOSITION II.2.** *For all  $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$ , the function  $\varphi_{\lambda, m}$  satisfies the product formula*

$$\varphi_{\lambda, m}(x, t) \varphi_{\lambda, m}(y, s) = T_{(x, t)}^{(\alpha)} \varphi_{\lambda, m}(y, s), \quad \text{for all } (x, t) \text{ and } (y, s) \text{ in } K. \quad (\text{II.2})$$

*Proof.* For  $\alpha > 0$  and  $x, y \in \mathbb{R}$ , from [10, p. 537] we have

$$\begin{aligned} \mathcal{L}_m^{(\alpha)}(x^2) \mathcal{L}_m^{(\alpha)}(y^2) &= \frac{2^\alpha}{\pi} \int_0^{2\pi} \int_0^1 \mathcal{L}_m^{(\alpha)}(x^2 + y^2 + 2xyr \cos \psi) \\ &\quad \times \cos(xyr \sin \psi) r(1 - r^2)^{\alpha-1} dr d\psi. \end{aligned} \quad (\text{II.3})$$

We deduce the result from this formula.

For  $\alpha = 0$  we obtain the result from [3, p. 79].

**COROLLARY II.1.** *For all  $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$ , the function  $\varphi_{\lambda, m}$  is infinitely differentiable on  $\mathbb{R}^2$ , even with respect to the first variable and satisfies*

$$\sup_{(x, t) \in K} |\varphi_{\lambda, m}(x, t)| = 1. \quad (\text{II.4})$$

**Remark II.2.** From the relation (II.4) we deduce that

$$\sup_{x \geq 0} |\mathcal{L}_m^{(\alpha)}(x)| = 1, \quad \text{for all } m \in \mathbb{N}. \quad (\text{II.5})$$

*Notation.* We denote by

- $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ .
- $F(\mathbb{R}^* \times \mathbb{N})$  the space of functions defined on  $\mathbb{R}^* \times \mathbb{N}$ .
- $\Delta_+$  and  $\Delta_-$  the operators defined on  $F(\mathbb{R}^* \times \mathbb{N})$  by

$$\begin{aligned} \Delta_+ g(\lambda, m) &= g(\lambda, m+1) - g(\lambda, m), \\ \Delta_- g(\lambda, m) &= \begin{cases} g(\lambda, m) - g(\lambda, m-1), & \text{if } m \geq 1 \\ g(\lambda, 0), & \text{if } m = 0. \end{cases} \end{aligned}$$

- $\Lambda_1$  and  $\Lambda_2$  the operators defined on  $F(\mathbb{R}^* \times \mathbb{N})$  by

$$\begin{aligned} \Lambda_1 g(\lambda, m) &= \frac{1}{|\lambda|} (m\Delta_+ \Delta_- g(\lambda, m) + (\alpha+1)\Delta_+ g(\lambda, m)), \\ \Lambda_2 g(\lambda, m) &= -\frac{1}{2\lambda} ((\alpha+m+1)\Delta_+ g(\lambda, m) + m\Delta_- g(\lambda, m)). \end{aligned}$$

**PROPOSITION II.3.** *For all  $(\lambda, m) \in \mathbb{R}^* \times \mathbb{N}$  and  $(x, t) \in K$ , we have*

$$-x^2 \varphi_{\lambda, m}(x, t) = \Lambda_1 \varphi_{\lambda, m}(x, t), \quad (\text{II.6})$$

$$it. \varphi_{\lambda, m}(x, t) = \left( \Lambda_2 + \frac{\partial}{\partial \lambda} \right) \varphi_{\lambda, m}(x, t). \quad (\text{II.7})$$

*Proof.* For all  $m \geq 1$ , from [2, p. 190] we have

$$\begin{aligned} xL_m^{(\alpha)}(x) &= (2m + \alpha + 1)L_m^{(\alpha)}(x) - (m + 1)L_{m+1}^{(\alpha)}(x) \\ &\quad - (m + \alpha)L_{m-1}^{(\alpha)}(x), \\ x \frac{d}{dx} L_m^{(\alpha)}(x) &= mL_m^{(\alpha)}(x) - (m + \alpha)L_{m-1}^{(\alpha)}(x). \end{aligned}$$

We deduce the results from these identities.

PROPOSITION II.4. Let  $f$  and  $g$  be in  $F(\mathbb{R}^* \times \mathbb{N})$  and satisfy

(i) For all  $\lambda \in \mathbb{R}^*$ , there exist  $k \in \mathbb{N}$  and  $C_{1,k,\lambda} > 0$  such that

$$\sup_{m \in \mathbb{N}} (1 + m^2)^{-k} |f(\lambda, m)| \leq C_{1,k,\lambda}.$$

(ii) For all  $\lambda \in \mathbb{R}^*$  and  $k \in \mathbb{N}$ , there exists  $C_{2,k,\lambda} > 0$  such that

$$\sup_{m \in \mathbb{N}} (1 + m^2)^k |g(\lambda, m)| \leq C_{2,k,\lambda}.$$

Then for all  $\lambda \in \mathbb{R}^*$ , we have

$$\sum_{m=0}^{+\infty} L_m^{(\alpha)}(0) f(\lambda, m) [\Lambda_1 g(\lambda, m)] = \sum_{m=0}^{+\infty} L_m^{(\alpha)}(0) [\Lambda_1 f(\lambda, m)] g(\lambda, m), \quad (\text{II.8})$$

$$\begin{aligned} &\sum_{m=0}^{+\infty} L_m^{(\alpha)}(0) f(\lambda, m) [\Lambda_2 g(\lambda, m)] + \sum_{m=0}^{+\infty} L_m^{(\alpha)}(0) [\Lambda_2 f(\lambda, m)] g(\lambda, m) \\ &= \sum_{m=0}^{+\infty} \frac{\alpha + 1}{\lambda} L_m^{(\alpha)}(0) f(\lambda, m) g(\lambda, m). \end{aligned} \quad (\text{II.9})$$

*Proof.* This is a consequence of the identity

$$(m + 1)L_{m+1}^{(\alpha)}(0) = (m + \alpha + 1)L_m^{(\alpha)}(0).$$

## 2. Generalized Fourier Transform on the Laguerre Hypergroup $K$

*Notations.* We denote by

- $\mathcal{E}_*(K)$  the space of continuous functions  $f: \mathbb{R}^2 \rightarrow \mathbb{C}$ , even with respect to the first variable.

- $\mathcal{S}_*(K)$  the space of functions  $f: \mathbb{R}^2 \rightarrow \mathbb{C}$ , even with respect to the first variable,  $\mathcal{E}^\infty$  on  $\mathbb{R}^2$  and rapidly decreasing together with their deriva-



tives, i.e., for all  $k, p, q \in \mathbb{N}$  we have

$$N_{k,p,q}(f) = \sup_{(x,t) \in K} \left\{ (1+x^2+t^2)^k \left| \frac{\partial^{p+q}}{\partial x^p \partial t^q} f(x,t) \right| \right\} < +\infty.$$

Equipped with the topology defined by the semi-norms  $N_{k,p,q}$ ,  $\mathcal{S}_*(K)$  is a Fréchet space.

- $\mathcal{D}_*(K)$  the space of functions  $f: \mathbb{R}^2 \rightarrow \mathbb{C}$ , even with respect to the first variable,  $\mathcal{E}^\infty$  on  $\mathbb{R}^2$  and with compact support.

- $\mathcal{S}(\mathbb{R} \times \mathbb{N})$  the space of functions  $g: \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{C}$ , such that

(i) For all  $m, p, q, r, s \in \mathbb{N}$ , the function

$$\lambda \rightarrow \lambda^p \left( |\lambda| \left( m + \frac{\alpha+1}{2} \right) \right)^q \Lambda_1^r \left( \Lambda_2 + \frac{\partial}{\partial \lambda} \right)^s g(\lambda, m)$$

is bounded and continuous on  $\mathbb{R}$ ,  $\mathcal{E}^\infty$  on  $\mathbb{R}^*$  such that the left and the right derivatives at zero exist.

(ii) For all  $k, p, q \in \mathbb{N}$ , we have

$$\nu_{k,p,q}(g)$$

$$= \sup_{(\lambda, m) \in \mathbb{R}^* \times \mathbb{N}} \left\{ (1+\lambda^2(1+m^2))^k \left| \Lambda_1^p \left( \Lambda_2 + \frac{\partial}{\partial \lambda} \right)^q g(\lambda, m) \right| \right\} < +\infty.$$

Equipped with the topology defined by the semi-norms  $\nu_{k,p,q}$ ,  $\mathcal{S}(\mathbb{R} \times \mathbb{N})$  is a Fréchet space.

- $\mathcal{D}(\mathbb{R} \times \mathbb{N})$  the subspace of  $\mathcal{S}(\mathbb{R} \times \mathbb{N})$  consisting of functions  $g$  satisfying

(i) There exists  $m_0 \in \mathbb{N}$  such that  $g(\lambda, m) = 0$  for all  $\lambda \in \mathbb{R}$  and  $m > m_0$ .

(ii) For all  $m \leq m_0$ , the function  $\lambda \rightarrow g(\lambda, m)$  is  $\mathcal{E}^\infty$  on  $\mathbb{R}$ , with compact support and vanishes in a neighborhood of zero.

- $L_\alpha^p(\mathbb{R} \times \mathbb{N})$ ,  $p \in [1, +\infty]$ , the space of measurable functions  $g: \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{C}$ , such that  $\|g\|_{L_\alpha^p} < +\infty$ , where

$$\|g\|_{L_\alpha^p} = \left[ \int_{\mathbb{R} \times \mathbb{N}} |g(\lambda, m)|^p d\gamma_\alpha(\lambda, m) \right]^{1/p}, \quad \text{if } p \in [1, +\infty[,$$

$$\|g\|_{L_\alpha^\infty} = \operatorname{ess\,sup}_{(\lambda, m) \in \mathbb{R} \times \mathbb{N}} |g(\lambda, m)|,$$

$d\gamma_\alpha$  being the positive measure defined on  $\mathbb{R} \times \mathbb{N}$  by

$$\int_{\mathbb{R} \times \mathbb{N}} g(\lambda, m) d\gamma_\alpha(\lambda, m) = \sum_{m=0}^{+\infty} L_m^{(\alpha)}(0) \int_{\mathbb{R}} g(\lambda, m) |\lambda|^{\alpha+1} d\lambda.$$

**DEFINITION II.1.** The generalized Fourier transform  $\mathcal{F}$  is defined on  $L^1_\alpha(K)$  by

$$\mathcal{F}(f)(\lambda, m) = \int_K \varphi_{-\lambda, m}(x, t) f(x, t) dm_\alpha(x, t), \quad (\lambda, m) \in \mathbb{R} \times \mathbb{N}.$$

**PROPOSITION II.5.** Let  $f$  be in  $L^1_\alpha(K)$ . Then

- (i) For all  $m \in \mathbb{N}$ , the function  $\lambda \rightarrow \mathcal{F}(f)(\lambda, m)$  is continuous on  $\mathbb{R}$ .
- (ii) The function  $\mathcal{F}(f)$  is bounded on  $\mathbb{R} \times \mathbb{N}$  and satisfies

$$\|\mathcal{F}(f)\|_{L^\infty_\alpha} \leq \|f\|_{\alpha, 1}. \quad (\text{II.10})$$

*Proof.* This is a consequence of Corollary II.1.

**PROPOSITION II.6.** (i) For all  $f$  and  $g$  in  $L^1_\alpha(K)$  we have

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g). \quad (\text{II.11})$$

(ii) Let  $f \in L^1_\alpha(K)$ . Then for all  $(x, t) \in K$  and  $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$ , we have

$$\mathcal{F}(T^{(\alpha)}_{(x, t)} f)(\lambda, m) = \varphi_{\lambda, m}(x, t) \mathcal{F}(f)(\lambda, m) \quad (\text{II.12})$$

*Proof.* The results follow from the product formula (II.2) and the commutativity of the convolution product.

**PROPOSITION II.7.** Let  $f \in \mathcal{E}_*(K)$ . Then  $f$  belongs to  $\mathcal{S}_*(K)$  if and only if

- (i) For all  $p, q \in \mathbb{N}$ , the function  $(x, t) \rightarrow D_1^p(D_2^q f)(x, t)$  is of class  $\mathcal{E}^2$  on  $\mathbb{R}^2$ .
- (ii) For all  $p, q, k \in \mathbb{N}$ , we have

$$\tilde{N}_{k, p, q}(f) = \sup_{(x, t) \in K} \left\{ (1 + x^2 + t^2)^k |D_1^p(D_2^q f)(x, t)| \right\} < +\infty.$$

*Proof.* We deduce the result from the following two lemmas

**LEMMA II.1.** Let  $f \in \mathcal{S}_*(K)$ , then the function  $(x, t) \rightarrow (1/x)(\partial f / \partial x)(x, t)$  belongs to  $\mathcal{S}_*(K)$ .

**LEMMA II.2.** For all  $f$  in  $\mathcal{E}_*(K)$  satisfying the assumptions (i) and (ii) of Proposition II.7, the function  $(x, t) \rightarrow (\partial f / \partial x)(x, t)$  is of class  $\mathcal{E}^2$  on  $K$  and

$$\frac{\partial f}{\partial x}(x, t) = x \int_0^1 D_3 f(xu, t) u^{2\alpha+1} du, \quad (x, t) \in \mathbb{R}^2,$$

where  $D_3$  is the differential operator defined by  $D_3 f(x, t) = (D_2 - x^2(\partial^2 / \partial t^2))f(x, t)$ .

*Proof.* The operator  $D_2$  can be written in the form

$$D_2 = \frac{1}{x^{2\alpha+1}} \frac{\partial}{\partial x} \left( x^{2\alpha+1} \frac{\partial}{\partial x} \right) + x^2 \frac{\partial^2}{\partial t^2}.$$

We deduce the result by using the Taylor formula.

*Remark II.3.* Equipped with the topology defined by the semi-norms  $\tilde{N}_{k,p,q}$ ,  $\mathcal{S}_*(K)$  is a Fréchet space. This topology is equivalent to that defined by the semi-norms  $N_{k,p,q}$ .

**PROPOSITION II.8.** *Let  $f$  be in  $\mathcal{S}_*(K)$ . Then*

(i) *For all  $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$ , we have*

$$\mathcal{F}(D_1 f)(\lambda, m) = i\lambda \mathcal{F}(f)(\lambda, m), \quad (\text{II.13})$$

$$\mathcal{F}(D_2 f)(\lambda, m) = -4|\lambda| \left( m + \frac{\alpha+1}{2} \right) \mathcal{F}(f)(\lambda, m). \quad (\text{II.14})$$

(ii) *For all  $m \in \mathbb{N}$ , the function  $\lambda \rightarrow \mathcal{F}(f)(\lambda, m)$  is continuous and bounded on  $\mathbb{R}$ ,  $\mathcal{C}^\infty$  on  $\mathbb{R}^*$  such that the left and the right derivatives at zero exist.*

(iii) *For all  $(\lambda, m) \in \mathbb{R}^* \times \mathbb{N}$ , we have*

$$\mathcal{F}(-x^2 f)(\lambda, m) = \Lambda_1 \mathcal{F}(f)(\lambda, m), \quad (\text{II.15})$$

$$\mathcal{F}(-itf)(\lambda, m) = \left( \Lambda_2 + \frac{\partial}{\partial \lambda} \right) \mathcal{F}(f)(\lambda, m). \quad (\text{II.16})$$

*Proof.* From Proposition II.1 we deduce the relations (II.13) and (II.14), and we obtain the relations (II.15) and (II.16) from Proposition II.3.

**COROLLARY II.2.** *The generalized Fourier transform  $\mathcal{F}$  is linear and continuous from  $\mathcal{S}_*(K)$  into  $\mathcal{S}(\mathbb{R} \times \mathbb{N})$ .*

**DEFINITION II.2.** We define the operator  $\mathcal{F}^{-1}$  on  $L_\alpha^1(\mathbb{R} \times \mathbb{N})$  by

$$\mathcal{F}^{-1}(g)(x, t) = \int_{\mathbb{R} \times \mathbb{N}} g(\lambda, m) \varphi_{\lambda, m}(x, t) d\gamma_\alpha(\lambda, m), \quad (x, t) \in K.$$

**PROPOSITION II.9.** *For all  $g \in L_\alpha^1(\mathbb{R} \times \mathbb{N})$ , the function  $\mathcal{F}^{-1}(g)$  is continuous on  $K$  and satisfies*

$$\|\mathcal{F}^{-1}(g)\|_{\alpha, \infty} \leq \|g\|_{L_\alpha^1}.$$

*Proof.* This is a consequence of Corollary II.1.

PROPOSITION II.10. Let  $g$  be in  $\mathcal{S}(\mathbb{R} \times \mathbb{N})$ . Then

(i) For all  $(x, t) \in K$ , we have

$$-it\mathcal{F}^{-1}(g)(x, t) = \mathcal{F}^{-1}\left(\left(\Lambda_2 + \frac{\partial}{\partial \lambda}\right)g\right)(x, t), \quad (\text{II.17})$$

$$-x^2\mathcal{F}^{-1}(g)(x, t) = \mathcal{F}^{-1}(\Lambda_1 g)(x, t). \quad (\text{II.18})$$

(ii) The function  $\mathcal{F}^{-1}(g)$  is of class  $\mathcal{C}^2$  on  $\mathbb{R}^2$ , even with respect to the first variable and satisfies

$$D_1\mathcal{F}^{-1}(g)(x, t) = \mathcal{F}^{-1}(i\lambda g)(x, t), \quad (x, t) \in K, \quad (\text{II.19})$$

$$D_2\mathcal{F}^{-1}(g)(x, t) = \mathcal{F}^{-1}\left(-4|\lambda|\left(m + \frac{\alpha+1}{2}\right)g\right)(x, t), \quad (x, t) \in K. \quad (\text{II.20})$$

*Proof.* (i) Let  $g$  be in  $\mathcal{S}(\mathbb{R} \times \mathbb{N})$ . For all  $m \in \mathbb{N}$ , we have

$$\begin{aligned} & \int_{\mathbb{R}} g(\lambda, m) \frac{\partial}{\partial \lambda} (\varphi_{\lambda, m}(x, t)) |\lambda|^{\alpha+1} d\lambda \\ &= -(\alpha+1) \int_{\mathbb{R}} \frac{g(\lambda, m)}{\lambda} \varphi_{\lambda, m}(x, t) |\lambda|^{\alpha+1} d\lambda \\ & \quad - \int_{\mathbb{R}} \frac{\partial g}{\partial \lambda}(\lambda, m) \varphi_{\lambda, m}(x, t) |\lambda|^{\alpha+1} d\lambda. \end{aligned}$$

From this equality together with (II.7) and (II.9) we deduce the identity (II.17). The identity (II.18) follows from the relations (II.6) and (II.8).

(ii) From [2, p. 189], we have for all  $m \geq 1$  and  $x \geq 0$

$$\frac{d}{dx} L_m^{(\alpha)}(x) = -L_{m-1}^{(\alpha+1)}(x).$$

By using the relation (II.5) we deduce that there are two nonnegative constants  $C_{1, \alpha}$  and  $C_{2, \alpha}$  such that for all  $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$  and  $x \geq 0$  we have

$$\begin{aligned} \left| \frac{d}{dx} \mathcal{L}_m^{(\alpha)}(|\lambda|x^2) \right| &\leq C_{1, \alpha} (1 + |x|) (1 + |\lambda|(1 + m)), \\ \left| \frac{d^2}{dx^2} \mathcal{L}_m^{(\alpha)}(|\lambda|x^2) \right| &\leq C_{2, \alpha} (1 + |x|)^2 (1 + |\lambda|(1 + m))^2. \end{aligned}$$

Hence we obtain the relations (II.19) and (II.20) from Proposition II.1.

LEMMA II.3. For all  $\lambda \in \mathbb{R}^*$ , the system

$$\left\{ \left( \frac{2|\lambda|^{\alpha+1} L_m^{(\alpha)}(0)}{\Gamma(\alpha+1)} \right)^{1/2} \mathcal{L}_m^{(\alpha)}(|\lambda|x^2) \right\}_{m \in \mathbb{N}}$$

forms an orthonormal basis of the space  $L^2([0, +\infty[, x^{2\alpha+1} dx)$  of square integrable functions on  $[0, +\infty[$  with respect to the measure  $x^{2\alpha+1} dx$  (see [15, p. 84]).

PROPOSITION II.11. The operator  $\mathcal{F}^{-1}$  is linear and continuous from  $\mathcal{S}(\mathbb{R} \times \mathbb{N})$  into  $\mathcal{S}_*(K)$  and we have

$$\mathcal{F}(\mathcal{F}^{-1}(g)) = g, \quad \text{for all } g \in \mathcal{S}(\mathbb{R} \times \mathbb{N}). \quad (\text{II.21})$$

*Proof.* From Propositions II.7, II.9 and II.10 we deduce that the operator  $\mathcal{F}^{-1}$  is linear and continuous from  $\mathcal{S}(\mathbb{R} \times \mathbb{N})$  into  $\mathcal{S}_*(K)$ .

Now by using the inversion formula for the classical Fourier transform we obtain

$$\frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}^{-1}(g)(x, t) e^{-i\lambda t} dt = \pi |\lambda|^{\alpha+1} \sum_{m=0}^{+\infty} L_m^{(\alpha)}(0) g(\lambda, m) \mathcal{L}_m^{(\alpha)}(|\lambda|x^2).$$

This gives the formula (II.21) by virtue of Lemma II.3.

PROPOSITION II.12. For all  $f \in L_{\alpha}^1(K) \cap L_{\alpha}^2(K)$ , we have the Plancherel formula

$$\|f\|_{\alpha, 2} = \|\mathcal{F}(f)\|_{L_{\alpha}^2}. \quad (\text{II.22})$$

*Proof.* Let  $f$  be in  $L_{\alpha}^1(K) \cap L_{\alpha}^2(K)$ . By the Plancherel theorem for the classical Fourier transform on  $\mathbb{R}$  we get

$$\|f\|_{\alpha, 2}^2 = \frac{1}{2\pi^2 \Gamma(\alpha+1)} \int_0^{+\infty} \left( \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x, t) e^{-i\lambda t} dt \right|^2 d\lambda \right) x^{2\alpha+1} dx. \quad (\text{II.23})$$

The formula (II.22) follows from this identity by using Lemma II.3.

THEOREM II.1. The generalized Fourier transform  $\mathcal{F}$  is a topological isomorphism from  $\mathcal{S}_*(K)$  onto  $\mathcal{S}(\mathbb{R} \times \mathbb{N})$ . Its inverse is the operator  $\mathcal{F}^{-1}$  given in Definition II.2.

*Proof.* The results follow from Corollary II.2 and Propositions II.11 and II.12.

PROPOSITION II.13. (i) *The space  $\mathcal{D}(\mathbb{R} \times \mathbb{N})$  is dense in  $L_\alpha^2(\mathbb{R} \times \mathbb{N})$ .*

(ii) *The space  $\mathcal{S}(\mathbb{R} \times \mathbb{N})$  is dense in  $L_\alpha^2(\mathbb{R} \times \mathbb{N})$ .*

*Proof.* Let  $g \in L_\alpha^2(\mathbb{R} \times \mathbb{N})$ . Then for any  $\varepsilon > 0$ , there exists  $m_0 \in \mathbb{N}$  such that

$$\int_{\mathbb{R} \times \mathbb{N}_{m_0}} |g(\lambda, m)|^2 d\gamma_\alpha(\lambda, m) < \varepsilon/2.$$

where  $\mathbb{N}_{m_0} = \{m_0 + 1, m_0 + 2, \dots\}$ .

For all  $m \in \{0, \dots, m_0\}$  we choose a  $\mathcal{C}^\infty$ -function  $G_m$  on  $\mathbb{R}$  with compact support, vanishing in a neighborhood of zero and satisfying

$$\int_{\mathbb{R}} |g(\lambda, m) - G_m(\lambda)|^2 |\lambda|^{\alpha+1} d\lambda < \varepsilon/2(m_0 + 1) \pi L_m^{(\alpha)}(0).$$

Let  $G$  be the function defined on  $\mathbb{R} \times \mathbb{N}$  by

$$G(\lambda, m) = \begin{cases} G_m(\lambda), & \text{if } m \in \{0, \dots, m_0\} \\ 0, & \text{if } m \in \mathbb{N}_{m_0}. \end{cases}$$

This function belongs to  $\mathcal{D}(\mathbb{R} \times \mathbb{N})$  and we have

$$\int_{\mathbb{R} \times \mathbb{N}} |g(\lambda, m) - G(\lambda, m)|^2 d\gamma_\alpha(\lambda, m) < \varepsilon,$$

which proves the assertion (i). The assertion (ii) follows from the assertion (i).

From Theorem II.1 and Propositions II.12 and II.13 we deduce the following theorem

THEOREM II.2 (Plancherel Theorem for  $\mathcal{F}$ ). *The generalized Fourier transform  $\mathcal{F}$  extends to an isometric isomorphism from  $L_\alpha^2(K)$  onto  $L_\alpha^2(\mathbb{R} \times \mathbb{N})$ .*

COROLLARY II.3. *For all  $f$  and  $g$  in  $L_\alpha^2(K)$  we have the following Parseval formula for the generalized Fourier transform  $\mathcal{F}$*

$$\int_K f(x, t) \bar{g}(x, t) dm_\alpha(x, t) = \int_{\mathbb{R} \times \mathbb{N}} \mathcal{F}(f)(\lambda, m) \overline{\mathcal{F}(g)(\lambda, m)} d\gamma_\alpha(\lambda, m). \quad (\text{II.24})$$

In [12], we establish the following inversion theorem for the generalized Fourier transform

**THEOREM II.3.** *For all  $f$  in  $L^1_\alpha(K)$  such that  $\mathcal{F}(f)$  belongs to  $L^1_\alpha(\mathbb{R} \times \mathbb{N})$ , we have the following inversion formula*

$$f(x, t) = \mathcal{F}^{-1}(\mathcal{F}(f))(x, t), \quad \text{a.e. on } K.$$

### III. THE RADON TRANSFORM ON $K$

**DEFINITION III.1.** For  $\alpha \geq 0$ , the Radon transform  $R_\alpha$  on the Laguerre hypergroup  $K$  is defined by

$$R_\alpha f(x, t) = \frac{2\pi^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^{+\infty} T_{(x,t)}^{(\alpha)} f(y, 0) y^{2\alpha+1} dy, \quad f \in \mathcal{S}_*(K), \quad (\text{III.1})$$

where the  $T_{(x,t)}^{(\alpha)}$ ,  $(x, t) \in K$ , are the generalized translation operators given in Definition I.1.

*Remark III.1.* In [17, p. 384] the Radon transform on the Heisenberg group  $\mathbb{H}^n$  is defined by

$$R(F)(z, t) = \int_{\mathbb{C}^n} F(\xi + z, t - \text{Im}(z_1 \overline{z_1'} + \cdots + z_n \overline{z_n'})) d\xi, \quad (z, t) \in \mathbb{H}^n,$$

where  $d\xi$  is the Lebesgue measure on  $\mathbb{C}^n \cong \mathbb{R}^{2n}$ .

If the function  $F$  is radial, i.e.,  $F(z, t) = f(\|z\|, t)$ , then the function  $R(F)$  is also radial and we have

$$R(F)(z, t) = R_{n-1}(f)(\|z\|, t). \quad (\text{III.2})$$

**PROPOSITION III.1.** *For all  $f \in \mathcal{S}_*(K)$ , we have*

$$\begin{aligned} R_\alpha(f)(x, t) &= \frac{2\pi^{\alpha+1/2}}{\Gamma(\alpha+1/2)} \\ &\times \int_0^{+\infty} \left[ \int_u^{+\infty} (v^2 - u^2)^{\alpha-1/2} \right. \\ &\times \{f(v, t+xu) + f(v, t-xu)\} v dv \Big] du. \quad (\text{III.3}) \end{aligned}$$

*Proof.* For  $\alpha = n - 1$ ,  $n$  being a positive integer, we put

$$F(z, t) = f(\|z\|, t), \quad (z, t) \in \mathbb{H}^n.$$

Thus from the relation (III.2) we find

$$R_{n-1}(f)(x, t) = R(F)(xe_1, t), \quad (x, t) \in K,$$

where  $e_1 = (1, 0, \dots, 0) \in \mathbb{C}^n$ . It follows that

$$R_{n-1}(f)(x, t) = \int_0^{+\infty} \left[ \int_{S^{2n-1}} f(y, t + xy \operatorname{Re}(\omega_1)) d\sigma_{2n-1}(\omega) \right] y^{2n-1} dy, \quad (\text{III.4})$$

where  $S^{2n-1} = \{\omega = (\omega_1, \dots, \omega_n) \in \mathbb{C}^n / |\omega_1|^2 + \dots + |\omega_n|^2 = 1\}$  is the unit sphere in  $\mathbb{C}^n$ , and  $d\sigma_{2n-1}$  is the surface measure on  $S^{2n-1}$ .

For  $n = 1$  the relation (III.4) can be written in the form

$$R_0(f)(x, t) = 2 \int_0^{+\infty} \left[ \int_0^\pi f(y, t + xy \cos \theta) d\theta \right] y dy. \quad (\text{III.5})$$

Now let  $n \geq 2$ , then for any function  $G: S^{2n-1} \rightarrow \mathbb{C}$ , integrable with respect to the measure  $\sigma_{2n-1}$  we have

$$\begin{aligned} & \int_{S^{2n-1}} G(\omega) d\sigma_{2n-1}(\omega) \\ &= \int_0^\pi \left[ \int_{S^{2n-2}} G(\cos \theta, (\sin \theta) \omega) d\sigma_{2n-2}(\omega) \right] (\sin \theta)^{2n-2} d\theta. \end{aligned}$$

Using this formula we get

$$\begin{aligned} R_{n-1}(f)(x, t) &= \frac{2\pi^{n-1/2}}{\Gamma(n-1/2)} \int_0^{+\infty} \\ &\quad \times \left[ \int_0^\pi f(y, t + xy \cos \theta) (\sin \theta)^{2(n-1)} d\theta \right] y^{2n-1} dy. \end{aligned} \quad (\text{III.6})$$

For  $\alpha = n - 1$ ,  $n$  being a positive integer, we obtain the formula (III.3) from the relations (III.5) and (III.6) by a change of variables. For general real  $\alpha \geq 0$ , we deduce this formula from the Carlson theorem by analytic continuation (see [18, p. 185]).



#### IV. RANGE OF SOME SUBSPACE OF $\mathcal{S}_*(K)$ BY THE RADON TRANSFORM $R_\alpha$

*Notations.* We denote by

- $\mathcal{S}_{*,1}(K)$  the subspace of  $\mathcal{S}_*(K)$ , consisting of functions  $g$  which vanish together with all their derivatives on  $\mathbb{R} \times \{0\}$ .
- $\mathcal{S}_{*,2}(K)$  the subspace of  $\mathcal{S}_*(K)$ , consisting of functions  $g$  such that

$$\int_{\mathbb{R}} t^k g(x, t) dt = 0, \quad \text{for all } k \in \mathbb{N} \text{ and all } x \in \mathbb{R}.$$

- $\mathcal{F}_1$  and  $\mathcal{F}_2$  the operators defined on  $L_\alpha^1(K)$  by

$$\mathcal{F}_1(f)(x, \lambda) = \int_{\mathbb{R}} f(x, t) e^{-i\lambda t} dt, \quad (x, \lambda) \in K,$$

$$\mathcal{F}_2(f)(\mu, \lambda) = \frac{2\pi^{\alpha+1}}{\Gamma(\alpha+1)} \int_K f(x, t) e^{-i\lambda t} j_\alpha(\mu x) x^{2\alpha+1} dx dt, \\ (\mu, \lambda) \in K,$$

where  $j_\alpha(x) = \Gamma(\alpha+1)(2/x)^\alpha J_\alpha(x)$ ,  $J_\alpha$  being the Bessel function of first kind and order  $\alpha$ .

PROPOSITION IV.1. *We have*

$$\begin{aligned} \mathcal{S}_{*,2}(K) &= \{f \in \mathcal{S}_*(K) / \mathcal{F}_1(f) \in \mathcal{S}_{*,1}(K)\} \\ &= \{f \in \mathcal{S}_*(K) / \mathcal{F}_2(f) \in \mathcal{S}_{*,1}(K)\}. \end{aligned}$$

*Proof.* The result is a consequence of the well-known properties of the ordinary Fourier transform on  $\mathbb{R}$  and the Fourier–Bessel transform (see [19, p. 148]).

PROPOSITION IV.2. *For all  $f \in \mathcal{S}_*(K)$ , we have*

$$\mathcal{F}_1(R_\alpha f)(x, \lambda) = \mathcal{F}_2(f)(x\lambda, \lambda), \quad (x, \lambda) \in K. \quad (\text{IV.1})$$

*Proof.* From the relation (III.3) we obtain

$$\begin{aligned} \mathcal{F}_1(R_\alpha f)(x, \lambda) &= \frac{2\pi^{\alpha+1/2}}{\Gamma(\alpha+1/2)} \int_0^{+\infty} \left( \int_{\mathbb{R}} \int_u^{+\infty} (v^2 - u^2)^{\alpha-1/2} (f(v, t+xu) \right. \\ &\quad \left. + f(v, t-xu)) e^{-i\lambda t} v dv dt \right) du. \end{aligned}$$

By a change of variables, this formula becomes

$$\begin{aligned}
 \mathcal{F}_1(R_\alpha f)(x, \lambda) &= \frac{2\pi^{\alpha+1/2}}{\Gamma(\alpha+1/2)} \int_0^{+\infty} \\
 &\quad \times \left( 2 \int_{\mathbb{R}} \int_u^{+\infty} (v^2 - u^2)^{\alpha-1/2} f(v, t) \cos(\lambda x u) e^{-i\lambda t} v \, dv \, dt \right) du \\
 &= \frac{4\pi^{\alpha+1/2}}{\Gamma(\alpha+1/2)} \int_{]0, +\infty[ \times \mathbb{R}} f(y, t) e^{-i\lambda t} \\
 &\quad \times \left( \int_0^y (y^2 - u^2)^{\alpha-1/2} \cos(\lambda x u) \, du \right) y \, dy \, dt.
 \end{aligned}$$

We deduce the relation (IV.1) by using the Poisson integral representation of the function  $j_\alpha$ , given by

$$j_\alpha(r\mu) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi} \cdot \Gamma(\alpha+1/2)} r^{-2\alpha} \int_0^r (r^2 - x^2)^{\alpha-1/2} \cos(\mu x) \, dx$$

(see [20])

*Remark IV.1.* For positive integer  $\alpha$ , the formula (IV.1) is a special case of 3.6 in [4, pp. 429–430].

**LEMMA IV.1.** *Let  $f$  be in  $\mathcal{S}_{*,1}(K)$ . Then for all  $a \in \mathbb{N}$  and  $b \in \mathbb{Z}$  the functions  $f_1$  and  $f_2$  defined on  $\mathbb{R} \times \mathbb{R}^*$  by*

$$f_1(\mu, \lambda) = (\mu\lambda)^a \lambda^b f(\mu\lambda, \lambda),$$

$$f_2(\mu, \lambda) = \left(\frac{\mu}{\lambda}\right)^a \lambda^b f\left(\frac{\mu}{\lambda}, \lambda\right)$$

*are bounded and*

$$\lim_{\lambda \rightarrow 0} f_1(\mu, \lambda) = \lim_{\lambda \rightarrow 0} f_2(\mu, \lambda) = 0, \quad \text{for all } \mu \in \mathbb{R}.$$

*Proof.* For  $b$  nonnegative the result is obvious. For  $b$  negative, by using the Taylor formula we obtain

$$\lambda^b f_1(\mu, \lambda) = \frac{1}{(-b-1)!} \int_0^1 (1-t)^{-b-1} \Delta_2^{-b} f(\mu\lambda, \lambda t) \, dt,$$

$$\lambda^b f_2(\mu, \lambda) = \frac{1}{(-b-1)!} \int_0^1 (1-t)^{-b-1} \Delta_2^{-b} f\left(\frac{\mu}{\lambda}, \lambda t\right) \, dt,$$

where  $\Delta_2$  is the partial differential operator given by  $\Delta_2 f(\mu, \lambda) = (\partial f / \partial \lambda)(\mu, \lambda)$ .

We deduce the result from the preceding two identities.

LEMMA IV.2. The operator  $\Phi$  defined by  $\Phi(f)(\mu, \lambda) = f(\mu\lambda, \lambda)$ , for  $(\mu, \lambda) \in \mathbb{R}^2$ , is a bijection from  $\mathcal{S}_{*,1}(K)$  onto itself. Its inverse  $\Phi^{-1}$  is given by

$$\Phi^{-1}(g)(\mu, \lambda) = \begin{cases} g\left(\frac{\mu}{\lambda}, \lambda\right), & \text{if } \lambda \neq 0 \\ 0, & \text{if } \lambda = 0. \end{cases}$$

*Proof.* For all  $f, g \in \mathcal{S}_{*,1}(K)$  and  $p, q, r, s$  positive integers we have

$$\mu^p \lambda^q \Delta_1^r \Delta_2^s \Phi(f)(\mu, \lambda) = \sum_{\substack{\text{finite} \\ b \in \mathbb{Z}; a, c, d \in \mathbb{N}}} C_{a,b,c,d} (\mu\lambda)^a \lambda^b (\Delta_1^c \Delta_2^d f)(\mu\lambda, \lambda), \quad (\text{IV.2})$$

$$\mu^p \lambda^q \Delta_1^r \Delta_2^s \Phi^{-1}(g)(\mu, \lambda) = \sum_{\substack{\text{finite} \\ \beta \in \mathbb{Z}; \alpha, \gamma, \delta \in \mathbb{N}}} C_{\alpha, \beta, \gamma, \delta} \left(\frac{\mu}{\lambda}\right)^\alpha \lambda^\beta (\Delta_1^\gamma \Delta_2^\delta g)\left(\frac{\mu}{\lambda}, \lambda\right), \quad (\text{IV.3})$$

where  $C_{a,b,c,d}$  and  $C_{\alpha, \beta, \gamma, \delta}$  are constants and  $\Delta_1$  is the partial differential operator given by  $\Delta_1 f(\mu, \lambda) = (\partial f / \partial \mu)(\mu, \lambda)$ .

The result follows from the relations (IV.2) and (IV.3) by using Lemma IV.1.

THEOREM IV.1. The Radon transform  $R_\alpha$  is a bijection from  $\mathcal{S}_{*,2}(K)$  onto itself.

*Proof.* Let  $f$  be in  $\mathcal{S}_{*,2}(K)$ . By Proposition IV.2 we have

$$\mathcal{F}_1(R_\alpha(f)) = \Phi(\mathcal{F}_2(f)).$$

From the second identity in Proposition IV.1 together with Lemma IV.2 we deduce that  $\mathcal{F}_1(R_\alpha(f))$  belongs to  $\mathcal{S}_{*,1}(K)$ . Since  $\mathcal{F}_1$  is a bijection from  $\mathcal{S}_*(K)$  onto itself it follows that  $R_\alpha(f)$  belongs to  $\mathcal{S}_*(K)$  and thus to  $\mathcal{S}_{*,2}(K)$  by virtue of the first identity in Proposition IV.1. This proves that  $R_\alpha(\mathcal{S}_{*,2}(K)) \subset \mathcal{S}_{*,2}(K)$ , and that for all  $f \in \mathcal{S}_{*,2}(K)$  we have

$$R_\alpha(f) = (\mathcal{F}_1^{-1} \circ \Phi \circ \mathcal{F}_2)(f).$$

From this relation together with Proposition IV.1, Lemma IV.2, and the fact that the transform  $\mathcal{F}_1$  (respectively  $\mathcal{F}_2$ ) is a bijection from  $\mathcal{S}_{*,2}(K)$  onto  $\mathcal{S}_{*,1}(K)$ , it follows that  $R_\alpha$  is a bijection from  $\mathcal{S}_{*,2}(K)$  onto itself. Its inverse  $R_\alpha^{-1}$  is given by

$$R_\alpha^{-1}(g) = (\mathcal{F}_2^{-1} \circ \Phi^{-1} \circ \mathcal{F}_1)(f).$$

This completes the proof of Theorem IV.1.

# V. INVERSION FORMULAS AND THE PLANCHEREL THEOREM FOR THE OPERATOR $R_\alpha$

*Notation.* For  $\alpha \geq 0$ , we denote by  $L_\alpha$  the operator defined on  $\mathcal{S}_*(K)$  by

$$L_\alpha f(x, t) = \begin{cases} \Delta_2^{\alpha+1} f(x, t), & \text{if } \alpha + 1 \in 2\mathbb{N} \\ I^{-(\alpha+1)} f(x, t), & \text{otherwise,} \end{cases}$$

where  $I^\gamma$  is the operator defined on  $\mathcal{S}_*(K)$  for  $\gamma \in \mathbb{C} \setminus (2\mathbb{N} + 1)$  by

$$I^\gamma f(x, t) = \frac{\Gamma((1 - \gamma)/2)}{\pi 2^\gamma \Gamma(\gamma/2)} \int_{\mathbb{R}} \frac{f(x, u)}{|t - u|^{1-\gamma}} du.$$

For  $\operatorname{Re}(\gamma) \leq 0$ , this is interpreted as an analytic continuation.

The space  $\mathcal{S}_{*,2}(K)$  is invariant under  $I^\gamma$  and we have for all  $f \in \mathcal{S}_*(K)$

$$\mathcal{F}_1 I^\gamma(f)(x, \lambda) = |\lambda|^{-2\gamma} \mathcal{F}_1(f)(x, \lambda), \quad (x, \lambda) \in K. \quad (\text{V.1})$$

(See [7, 8, p. 67].)

**PROPOSITION V.1.** *Let  $f$  be in  $\mathcal{S}_{*,2}(K)$ . Then  $L_\alpha R_\alpha(f)$  belongs to  $\mathcal{S}_{*,2}(K)$  and*

$$\mathcal{F}(L_\alpha R_\alpha f)(\lambda, m) = (2\pi)^{\alpha+1} (-1)^m \mathcal{F}(f)(\lambda, m), \quad (\lambda, m) \in \mathbb{R} \times \mathbb{N}, \quad (\text{V.2})$$

where  $\mathcal{F}$  is the generalized Fourier transform given in Definition II.1.

*Proof.* Using the relation (IV.1) we obtain

$$\begin{aligned} \mathcal{F}(R_\alpha f)(\lambda, m) &= \frac{2\pi^\alpha}{[\Gamma(\alpha + 1)]^2} \int_{]0, +\infty[ \times \mathbb{R}} f(y, t) e^{-i\lambda t} \\ &\quad \times \left[ \int_0^{+\infty} \mathcal{L}_m^{(\alpha)}(|\lambda|x^2) j_\alpha(\lambda xy) x^{2\alpha+1} dx \right] y^{2\alpha+1} dy. \end{aligned}$$

Now from [3, p. 92] we have

$$x^{\alpha/2} e^{-x} L_m^{(\alpha)}(x) = \frac{1}{m!} \int_0^{+\infty} t^{m+\alpha/2} J_\alpha(2\sqrt{xt}) e^{-t} dt.$$

Thus

$$|\lambda|^{\alpha+1} \mathcal{L}_m^{(\alpha)}(|\lambda|x^2) = \frac{(-1)^m}{2^\alpha \Gamma(\alpha + 1)} \int_0^{+\infty} j_\alpha(|\lambda|xt) \mathcal{L}_m^{(\alpha)}(|\lambda|t^2) t^{2\alpha+1} dt.$$

It follows that

$$|\lambda|^{\alpha+1} \mathcal{F}(R_\alpha f)(\lambda, m) = (-1)^m (2\pi)^{\alpha+1} \mathcal{F}(f)(\lambda, m),$$

$$(\lambda, m) \in \mathbb{R}^* \times \mathbb{N}. \quad (\text{V.3})$$

On the other hand, it is easy to see that for  $(\alpha + 1) \in 2\mathbb{N}$  and  $g \in \mathcal{S}_{*,2}(K)$  we have

$$\mathcal{F}_1(\Delta_2^{\alpha+1} g)(x, \lambda) = |\lambda|^{\alpha+1} \mathcal{F}_1(g)(x, \lambda), \quad (x, \lambda) \in K.$$

From this equality together with the relation (V.1) we deduce

$$\mathcal{F}_1(L_\alpha(g))(x, \lambda) = |\lambda|^{\alpha+1} \mathcal{F}_1(g)(x, \lambda), \quad (x, \lambda) \in K.$$

But for all  $h \in \mathcal{S}_*(K)$  we have

$$\mathcal{F}(h)(\lambda, m) = \frac{1}{\pi \Gamma(\alpha + 2)} \int_0^{+\infty} \mathcal{F}_1(h)(x, \lambda) \mathcal{L}_m^{(\alpha)}(|\lambda|x^2) x^{2\alpha+1} dx.$$

Then

$$\mathcal{F}(L_\alpha(g))(\lambda, m) = |\lambda|^{\alpha+1} \mathcal{F}(g)(\lambda, m), \quad \text{for all } g \in \mathcal{S}_{*,2}(K). \quad (\text{V.4})$$

We deduce the result from the relations (V.3) and (V.4).

**COROLLARY V.1.** *For all  $f$  in  $\mathcal{S}_{*,2}(K)$  we have*

$$L_\alpha R_\alpha(f) = R_\alpha L_\alpha(f). \quad (\text{V.5})$$

*Proof.* Since  $\mathcal{S}_{*,2}(K)$  is invariant under  $L_\alpha$ , then from the relations (V.2) and (V.4), we have for all  $f \in \mathcal{S}_{*,2}(K)$

$$\begin{aligned} |\lambda|^{\alpha+1} \mathcal{F}(R_\alpha L_\alpha(f))(\lambda, m) &= |\lambda|^{\alpha+1} \mathcal{F}(L_\alpha R_\alpha(f))(\lambda, m) \\ &= |\lambda|^{\alpha+1} (-1)^m (2\pi)^{\alpha+1} \mathcal{F}(f)(\lambda, m). \end{aligned}$$

We deduce the formula (V.5) by using the injectivity of the transform  $\mathcal{F}$  on  $\mathcal{S}_*(K)$ .

**THEOREM V.1.** *For all  $f \in \mathcal{S}_{*,2}(K)$ , we have the inversion formulas*

$$(L_\alpha R_\alpha)^2 f = (2\pi)^{2(\alpha+1)} f, \quad (\text{V.6})$$

$$R_\alpha (L_\alpha)^2 R_\alpha f = (2\pi)^{2(\alpha+1)} f, \quad (\text{V.7})$$

where  $(L_\alpha R_\alpha)^2 = L_\alpha R_\alpha L_\alpha R_\alpha$  and  $(L_\alpha)^2 = L_\alpha L_\alpha$ .

*Proof.* From the injectivity of the operator  $\mathcal{F}$  on  $\mathcal{S}_*(K)$  we deduce the formula (V.6). The formula (V.7) follows from the relations (V.5) and (V.6).

*Remark V.1.* For positive integer  $\alpha$ , Theorem V.1 is a special case of 4.1 and 4.2 in [4, pp. 436–437].

**PROPOSITION V.2.** *The operator  $L_\alpha R_\alpha$  is an isomorphism from  $\mathcal{S}_{*,2}(K)$  onto itself. Moreover, we have the Plancherel formula*

$$\int_K |f(x, t)|^2 dm_\alpha(x, t) = (2\pi)^{-2(\alpha+1)} \int_K |L_\alpha R_\alpha f(x, t)|^2 dm_\alpha(x, t),$$

for  $f \in \mathcal{S}_{*,2}(K)$ . (V.8)

*Proof.* By virtue of Theorem V.1 we have

$$(L_\alpha R_\alpha)^2(f) = f, \quad \text{for all } f \in \mathcal{S}_{*,2}(K).$$

From this identity together with Theorem IV.1 and the fact that  $\mathcal{S}_{*,2}(K)$  is invariant under the operator  $L_\alpha$  we deduce that  $L_\alpha R_\alpha$  is a bijection from  $\mathcal{S}_{*,2}(K)$  onto itself and that

$$(L_\alpha R_\alpha)^{-1} = L_\alpha R_\alpha = R_\alpha L_\alpha.$$

The formula (V.8) follows from the relation (V.2) by using the Plancherel formula (II.22) for the generalized Fourier transform  $\mathcal{F}$ .

Using the density of  $\mathcal{S}_{*,2}(K)$  in the space  $L_\alpha^2(K)$ , we deduce from Proposition V.2 the following theorem

**THEOREM V.2** (Plancherel Theorem for  $R_\alpha$ ). *The operator  $(2\pi)^{-(\alpha+1)} L_\alpha R_\alpha$  extends to an isometric isomorphism from  $L_\alpha^2(K)$  onto itself. Furthermore, we have*

$$(L_\alpha R_\alpha)^2 f = (2\pi)^{2(\alpha+1)} f, \quad \text{for all } f \in L_\alpha^2(K).$$

*Remark V.2.* For positive integer  $\alpha$ , Theorem V.2 is a special case of Theorem 3.16 in [4, p. 434].

## VI. INVERSION OF THE RADON TRANSFORM $R_\alpha$ BY USING GENERALIZED WAVELETS

In this section we define generalized wavelets and generalized continuous wavelet transforms on the Laguerre hypergroup  $K$  and we establish an inversion formula for these transforms. (For other results on these transforms one can see [12].)

As application of the generalized wavelets on  $K$  we establish a formula which gives the inverse operator of the Radon transform  $R_\alpha$  on the Laguerre hypergroup  $K$ .

DEFINITION VI.1. Let  $g \in L_\alpha^2(K)$ . We say that  $g$  is a generalized wavelet on  $K$  if there is a constant  $C_g \in ]0, +\infty[$  such that for all  $m \in \mathbb{N}$  and  $\lambda \in \mathbb{R}^*$ , we have

$$C_g = \int_{\mathbb{R}} |\mathcal{F}(g)(a\lambda, m)|^2 \frac{da}{|a|}. \quad (\text{VI.1})$$

EXAMPLE. For  $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$ , we put

$$\theta(\lambda, m) = \lambda^2 \left( m + \frac{\alpha + 1}{2} \right)^2 e^{-\lambda^2(m + (\alpha + 1)/2)^2}.$$

The function  $g = \mathcal{F}^{-1}(\theta)$  is a generalized wavelet on  $K$ .

PROPOSITION VI.1. Let  $g \in L_\alpha^2(K)$  be a generalized wavelet on  $K$ . Then for all  $a \in \mathbb{R}^*$ , the function  $(\lambda, m) \rightarrow \mathcal{F}(g)(a\lambda, m)$  belongs to  $L_\alpha^2(\mathbb{R} \times \mathbb{N})$  and we have

$$\left( \int_{\mathbb{R}} |\mathcal{F}(g)(a\lambda, m)|^2 d\gamma_\alpha(\lambda, m) \right)^{1/2} = |a|^{-(1+\alpha/2)} \|g\|_{\alpha, 2}. \quad (\text{VI.2})$$

*Proof.* By using a change of variables we deduce the identity (VI.2) from the Plancherel Theorem II.2 for the generalized Fourier transform  $\mathcal{F}$ .

Remark VI.1. For all  $g \in \mathcal{S}_*(K)$  we have

$$\mathcal{F}(g_a)(\lambda, m) = \mathcal{F}(g)(a\lambda, m), \quad \text{a.e. on } \mathbb{R} \times \mathbb{N}, \quad (\text{VI.3})$$

where  $g_a$  is the function defined by

$$g_a(x, t) = \frac{1}{|a|^{\alpha+2}} g\left(\frac{x}{\sqrt{|a|}}, \frac{t}{a}\right), \quad \text{a.e. on } K.$$

DEFINITION VI.2. Let  $g \in \mathcal{S}_*(K)$  be a generalized wavelet on  $K$ . The generalized continuous wavelet transform  $\Phi_g$  is defined on  $\mathcal{S}_*(K)$  by

$$\begin{aligned} \Phi_g(f)(a, (x, t)) &= \int_K f(y, s) \bar{g}_{a, (y, t)}(y, -s) dm_a(y, s), \\ (a, (x, t)) &\in \mathbb{R}^* \times K, \end{aligned} \quad (\text{VI.4})$$

where

$$g_{a, (x, t)}(y, s) = |a|^{1+\alpha/2} T_{(x, t)}^{(\alpha)} g_a(y, s), \quad \text{for } (y, s) \in K,$$

$T_{(x, t)}^{(\alpha)}$  being the generalized translation operators given in Definition I.1.

*Remark VI.2.* The relation (VI.4) can also be written as

$$\Phi_g(f)(a, (x, t)) = |a|^{1+\alpha/2} (f * \bar{g}_a)(x, t), \quad (\text{VI.5})$$

where  $*$  is the convolution product given by the relation (I.1).

**THEOREM VI.1.** *Let  $g \in \mathcal{S}_*(K)$  be a generalized wavelet on  $K$ . For all  $f$  in  $\mathcal{S}_*(K)$ , we have the following inversion formula*

$$f(x, t) = \frac{1}{C_g} \int_{\mathbb{R} \times K} \Phi_g(f)(a, (y, s)) g_{a, (y, s)}(x, -t) dm_\alpha(y, s) \frac{da}{|a|^{\alpha+3}}, \quad (x, t) \in K. \quad (\text{VI.6})$$

*Proof.* From the Parseval formula (II.24) for the Fourier transform  $\mathcal{F}$ , we deduce

$$\begin{aligned} & \int_K \Phi_g(f)(a, (y, s)) g_{a, (y, s)}(x, -t) dm_\alpha(y, s) \\ &= |a|^{\alpha+2} \int_{\mathbb{R} \times \mathbb{N}} \mathcal{F}(f * \bar{g}_a)(\lambda, m) \mathcal{F}(T_{(x, -t)}^{(\alpha)} g_a)(-\lambda, m) d\gamma_\alpha(\lambda, m). \end{aligned}$$

Now by Proposition II.6 we have

$$\begin{aligned} \mathcal{F}(f * \bar{g}_a)(\lambda, m) &= \mathcal{F}(f)(\lambda, m) \mathcal{F}(\bar{g}_a)(\lambda, m), \quad (\lambda, m) \in \mathbb{R} \times \mathbb{N}, \\ \mathcal{F}(T_{(x, -t)}^{(\alpha)} g_a)(-\lambda, m) &= \varphi_{-\lambda, m}(x, -t) \mathcal{F}(g_a)(-\lambda, m), \\ & \quad (\lambda, m) \in \mathbb{R} \times \mathbb{N}. \end{aligned}$$

But for all  $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$  we have

$$\mathcal{F}(\bar{g}_a)(\lambda, m) = \overline{\mathcal{F}(g_a)}(-\lambda, m) \quad \text{and} \quad \varphi_{-\lambda, m}(x, -t) = \varphi_{\lambda, m}(x, t).$$

It follows that

$$\begin{aligned} & \int_K \Phi_g(f)(a, (y, s)) g_{a, (y, s)}(x, -t) dm_\alpha(y, s) \\ &= |a|^{\alpha+2} \int_{\mathbb{R} \times \mathbb{N}} \mathcal{F}(f)(\lambda, m) |\mathcal{F}(g_a)(-\lambda, m)|^2 \varphi_{\lambda, m}(x, t) d\gamma_\alpha(\lambda, m). \end{aligned}$$

Integrating both sides of this equality over  $\mathbb{R}$  with respect to the measure  $da/|a|^{\alpha+3}$  we get

$$\begin{aligned} & \frac{1}{C_g} \int_{\mathbb{R}} \left[ \int_K \Phi_g(f)(a, (y, s)) g_{a, (y, s)}(x, -t) dm_\alpha(y, s) \right] \frac{da}{|a|^{\alpha+3}} \\ &= \int_{\mathbb{R} \times \mathbb{N}} \mathcal{F}(f)(\lambda, m) \varphi_{\lambda, m}(x, t) d\gamma_\alpha(\lambda, m) \\ &= \mathcal{F}^{-1}(\mathcal{F}(f))(x, t). \end{aligned}$$

We deduce the result by using Theorem II.1.



*Remark VI.3.* In [12], we show that the inversion formula (VI.6) holds almost everywhere on  $K$  for all  $f \in L^1_\alpha(K)$  such that  $\mathcal{F}(f) \in L^1_\alpha(\mathbb{R} \times \mathbb{N})$ .

**PROPOSITION VI.2.** *Let  $f$  and  $g$  be in  $\mathcal{S}_{*,2}(K)$ , then for all  $a \in \mathbb{R}^*$  we have*

$$(L_\alpha R_\alpha(g))_a = L_\alpha R_\alpha(g_a). \quad (\text{VI.7})$$

$$\tilde{\Phi}_{(L_\alpha(g))_a}(f) = |a|^{\alpha+1} \tilde{\Phi}_{g_a}(L_\alpha(f)), \quad (\text{VI.8})$$

where  $\tilde{\Phi}_h$ ,  $h \in \mathcal{S}_*(K)$ , denotes the transform defined on  $\mathcal{S}_*(K)$  by

$$\tilde{\Phi}_h(f)(x, t) = (f * \bar{h})(x, t).$$

*Proof.* From the first identity in Proposition II.6 we deduce that for all  $a \in \mathbb{R}^*$  we have

$$\mathcal{F}[\tilde{\Phi}_{(L_\alpha(g))_a}(f)] = |a|^{\alpha+1} \mathcal{F}(f) \mathcal{F}[\overline{L_\alpha(g_a)}], \quad (\text{VI.9})$$

$$\mathcal{F}[\tilde{\Phi}_{g_a}(L_\alpha(f))] = \mathcal{F}[L_\alpha(f)] \mathcal{F}(\bar{g}_a). \quad (\text{VI.10})$$

We obtain the formula (VI.7) (resp. (VI.8)) from the relations (V.2), (VI.3) (resp. (VI.9), (VI.10), (V.4)) by using the injectivity of the generalized Fourier transform on  $\mathcal{S}_*(K)$ .

**THEOREM VI.2.** *Let  $g \in \mathcal{S}_{*,2}(K)$  be a generalized wavelet on  $K$ , then for all  $f \in \mathcal{S}_{*,2}(K)$ , we have*

$$|\alpha|^{-\alpha/2} \tilde{\Phi}_{(L_\alpha R_\alpha L_\alpha(g))_a}(R_\alpha(f))(x, t) = (2\pi)^{2(\alpha+1)} \Phi_g(f)(a, (x, t)),$$

$$(a, (x, t)) \in \mathbb{R}^* \times K. \quad (\text{VI.11})$$

*Proof.* From the relation (VI.8), we deduce

$$\tilde{\Phi}_{(L_\alpha R_\alpha L_\alpha(g))_a}(R_\alpha(f)) = |a|^{\alpha+1} \tilde{\Phi}_{(R_\alpha L_\alpha(g))_a}(L_\alpha R_\alpha(f)), \quad \text{for all } a \in \mathbb{R}^*.$$

Applying the Fourier transform and using the identities (V.7) and (VI.9) we obtain

$$\mathcal{F}[\tilde{\Phi}_{(L_\alpha R_\alpha L_\alpha(g))_a}(R_\alpha(f))] = |a|^{\alpha+1} \mathcal{F}(L_\alpha R_\alpha(f)) \mathcal{F}[(\overline{L_\alpha R_\alpha(g)})_a],$$

for all  $a \in \mathbb{R}^*$ .

From this equality together with (VI.7) and (V.2) we get

$$\mathcal{F}[\tilde{\Phi}_{(L_\alpha R_\alpha L_\alpha(g))_a}(R_\alpha(f))] = (2\pi)^{2(\alpha+1)} |a|^{\alpha+1} \mathcal{F}(f) \mathcal{F}(\bar{g}_a). \quad (\text{VI.12})$$

But from the first identity in Proposition II.6 we have

$$\mathcal{F}[\tilde{\Phi}_{g_a}(f)] = \mathcal{F}(f) \mathcal{F}(\bar{g}_a). \quad (\text{VI.13})$$

We obtain the result from the relations (VI.12) and (VI.13) by using the injectivity of the generalized Fourier transform on  $\mathcal{S}_{*,2}(K)$ .

From Theorems (VI.1) and (VI.2) we deduce the following theorem which gives the expression of the inverse  $R_\alpha^{-1}$  of the operator  $R_\alpha$ , by means of the generalized wavelets.

**THEOREM VI.3.** *Let  $g \in \mathcal{S}_{*,2}(K)$  be a generalized wavelet on  $K$ . Then for all  $f$  in  $\mathcal{S}_{*,2}(K)$  we have*

$$R_\alpha^{-1}(f)(x, t) = \frac{(2\pi)^{-2\alpha-2}}{C_g} \int_{\mathbb{R} \times K} \tilde{\Phi}_{(L_\alpha R_\alpha L_\alpha(g))_a}(f)(y, s) \\ g_{a, (y, s)}(x, -t) dm_\alpha(y, s) \frac{da}{|a|^{3(\alpha/2+1)}}, \quad (x, t) \in K.$$

**Remark VI.4.** The function  $L_\alpha R_\alpha L_\alpha(g)$  is not necessarily a generalized wavelet on the Laguerre hypergroup  $K$ . But, if we suppose that  $L_\alpha(g)$  is a generalized wavelet together with  $g$  then by the relation (V.2) the function  $L_\alpha R_\alpha L_\alpha(g)$  becomes also a generalized wavelet.

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